THE INJECTIVITY OF THE POMPEIU TRANSFORM AND L^p-ANALOGUES OF THE WIENER–TAUBERIAN THEOREM

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ABSTRACT

Let E be a bounded Borel subset of $\mathbb{R}^n$, $n \geq 2$, of positive Lebesgue measure and P_E the corresponding 'Pompeiu transform'. We prove that P_E is injective on $L^p(\mathbb{R}^n)$ if $1 \leq p \leq 2n/(n-1)$. We explore the connection between this problem and a Wiener–Tauberian type theorem for the $M(n)$ action on $L^q(\mathbb{R}^n)$ for various values of q. We also take up the question of when P_E is injective in case E is of finite, positive measure, but is not necessarily a bounded set. Finally, we briefly look at these questions in the contexts of symmetric spaces of compact and non-compact type.

1. Introduction

This note is motivated by a result due to Thangavelu [12] that the spherical mean value operator T_r (of a fixed radius $r > 0$) on $L^p(R^n)$ is injective if $1 \leq p \leq 2n/(n-1)$. (See also [1] and [2].) In fact, this result is related to the following question about the 'Pompeiu transform' (see Section 2 for terminology): For an arbitrary bounded Borel subset E of $\mathbb{R}^n$ of positive Lebesgue measure, on what spaces can we assert that the Pompeiu transform P_E is injective? For instance, it is proved in [5] that if the set E is of the form $E_1 \times E_2 \times \cdots \times E_n$, then P_E is injective on $C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at ∞. However, it is shown in [9] that P_E is never injective on $C_0(\mathbb{R}^n)$ if E is spherically symmetric. The example given in [9] for $f \in \mathrm{Ker} P_E$ belongs to $C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for every

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$p > 2n/(n-1)$. On the other hand, it is easy to show that P_E is injective on $L^p(\mathbb{R}^n)$ for $1 \le p \le 2$ (and hence on $C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for such p). Thus it is natural to ask: In general, what decay conditions on $f \in C_0(\mathbb{R}^n)$ will force $f \equiv 0$ when $P_E f = 0$? Our main result implies that if $f \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $1 \le p \le 2n/(n-1)$, then indeed $f \equiv 0$ if $P_E f = 0$ and, in view of what has been said earlier, this is the best possible result in general (i.e. without assuming anything about the 'shape' of E).

For a survey of problems of the Pompeiu type see [3] or [13].

2. Notation, terminology and preliminary results

Throughout this section n will be greater than or equal to 2. Most of the notation and terminology we follow is fairly standard — see, for example, [7]. $\mathcal{D}(\mathbb{R}^n)$ will denote the space of C^∞-functions of compact support, $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions, $\mathcal{E}'(\mathbb{R}^n)$ the space of compactly supported distributions, $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. For each $\lambda > 0$, define ϕ_λ as follows:

$$\phi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} d\omega, \quad x \in \mathbb{R}^n.$$

Here (,) denotes the usual inner product, S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $d\omega$ is the canonical (probability) measure on S^{n-1}. Define $\phi_{\lambda,k}$ by

$$\phi_{\lambda,k}(x) = \frac{d^k}{d\lambda^k}\phi_\lambda(x).$$

(Thus $\phi_{\lambda,0} = \phi_\lambda$.) These functions can be explicitly written down in terms of well-known Bessel functions.

Note that if $f \in L^p(\mathbb{R}^n), 1 \le p \le \infty$, then f can be viewed as a tempered distribution and hence $\hat{f}$, also written as $f^\wedge$ (its Fourier transform), makes sense as a tempered distribution. If T is a radial tempered distribution, then so is $\hat{T}$ and if further T is compactly supported, then $\hat{T}$ is a smooth function and $\hat{T}(v) = T(\phi_{\|v\|})$. For a distribution T, Supp T will denote the (closed) support of T. For any function g, Z_g denotes the set $\{x : g(x) = 0\}$. If g is a continuous function on $\mathbb{R}^n$ define $g^\#$ by $g^\#(x) = \int_{\mathrm{SO}(n)} g(kx)dk, \quad x \in \mathbb{R}^n$. Here $\mathrm{SO}(n)$ is the special orthogonal group and dk the normalized Haar measure on $\mathrm{SO}(n)$. $g^\#$ is a continuous radial function and if $g \in L^p(\mathbb{R}^n), g^\#$ is also in $L^p(\mathbb{R}^n)$. For $\alpha > 0$, let M_α denote the sphere $\{v \in \mathbb{R}^n \colon \|v\| = \alpha\}$.

Next we record three lemmas that will be needed in the next section.

LEMMA 2.0: *Let T be a non-trivial radial distribution of compact support such that $\hat{T}$ vanishes on M_α, for some $\alpha > 0$. Then there exists an annulus $A_\epsilon = \{x \in \mathbb{R}^n : \alpha - \epsilon < \|x\| < \alpha + \epsilon\}, \epsilon > 0$, such that $\hat{T}$ has no other zeros in this annulus.*

(Let $h(\lambda) = \hat{T}(v)$ where $\|v\| = \lambda$. Then from the compactness of the support of T, it follows easily that h extends to an even, entire function of λ and the lemma follows easily from this observation.)

The next lemma follows easily from the asymptotic behaviour of Bessel functions. (This result is also used in the work of Thangavelu — see the proof of Theorem 2.2 in [12].)

LEMMA 2.1: *Let $\lambda > 0$ and let $f = \sum_{k=0}^{N} a_k \phi_{\lambda,k}$. If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2n/(n-1)$, then $a_0 = a_1 = \cdots = a_N = 0$.*

Finally, we record the following easy fact about radial distributions:

LEMMA 2.2: *Let T be a tempered radial distribution. Suppose $\operatorname{Supp} \hat{T} = M_\lambda$, for some $\lambda > 0$. Then $T = \sum_{k=0}^{N} a_k \phi_{\lambda,k}$, for some constants $a_0, a_1, \ldots, a_N$.*

(It is easy to see that the Fourier transform of $\sum_{k=0}^{N} a_k \phi_{\lambda,k}$ considered as a tempered distribution is supported on M_λ. The above lemma is the converse and follows from standard Fourier Analysis and Distribution Theory.)

Finally we come to the definition of the Pompeiu transform: Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and E be a bounded Borel set of positive Lebesgue measure. Then the 'Pompeiu transform' of f, denoted by $P_E f$, is a function on the group $M(n)$ of proper rigid motions of $\mathbb{R}^n$, defined by $P_E f(\sigma) = \int_{\sigma(E)} f$, where the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. (Recall that $\sigma \in M(n)$ if and only if $\sigma(v) = T(v) + v_0$ for some $T \in \mathrm{SO}(n)$ and $v_0 \in \mathbb{R}^n$.)

3. The Pompeiu transform

Throughout this section n is greater than or equal to 2. We first start with a generalization of Thangavelu's result from which our main result will be deduced:

PROPOSITION 3.1: *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ be non-trivial and radial. Let $1 \leq p \leq 2n/(n-1)$. If $f \in L^p(\mathbb{R}^n)$ and $f * T = 0$, then $f = 0$ a.e.*

Proof: By convolving f against an approximate identity, if necessary, we can assume that f is continuous or even smooth. Suppose now $f \not\equiv 0$. We will show this leads to a contradiction. Since translates of f are also 'killed' by convolution against T, we may assume that $f(0) \neq 0$. Hence $f^{\#}$ is also a continuous function, $f^{\#}(0) \neq 0$, and since T is radial, one can easily verify that $f^{\#} * T = 0$. Also, $f^{\#}$ is in $L^p(\mathbb{R}^n)$. Thus we may assume, by replacing f by $f^{\#}$, that f is a non-trivial, continuous, radial function (in $L^p(\mathbb{R}^n)$). Since f is non-trivial, $\hat{f}$ is a non-trivial tempered distribution. Hence Supp $\hat{f}$ is non-empty. Also, there exists $0 \neq v_0 \in \operatorname{Supp} \hat{f}$. (Otherwise, if $\operatorname{Supp} \hat{f} = \{0\}$, it will follow that f is a non-trivial polynomial and this contradicts the fact that $f \in L^p(\mathbb{R}^n)$ and $p < \infty$.) Since $\hat{f}$ is also radial, if $\alpha = \|v_0\|$, $M_\alpha \subseteq \operatorname{Supp} \hat{f}$. Since T is compact, $\hat{T}$ is given by a smooth function. Thus $(f * T)^\wedge = \hat{T}\hat{f}$ and since $(f * T)^\wedge = 0$, it follows that $\operatorname{Supp} \hat{f} \subseteq Z_{T^\wedge}$. Thus $M_\alpha \subseteq \operatorname{Supp} \hat{f} \subseteq Z_{T^\wedge}$. By Lemma 2.0, there exists $\epsilon > 0$ such that the only zeros of $\hat{T}$ in the annulus $A_\epsilon = \{v \in \mathbb{R}^n : \alpha - \epsilon < \|v\| < \alpha + \epsilon\}$ lie on M_α. Thus we have $\operatorname{Supp} \hat{f} \cap A_\epsilon = M_\alpha$. Now choose a non-trivial, radial $\psi \in \mathcal{D}$ which is 1 in a neighbourhood of M_α and zero outside $A_{\frac{\epsilon}{2}}$. Then $\psi\hat{f}$ is a non-trivial radial distribution and Supp $\psi\hat{f} = M_\alpha$. But $\psi\hat{f} = (\hat{\psi} * f)^\wedge$ and $\hat{\psi} * f$ is therefore non-trivial. Further, it is in $L^p(\mathbb{R}^n)$. (Since $\psi \in \mathcal{D}, \hat{\psi} \in \mathcal{S}$ and hence $\hat{\psi} * f \in L^p(\mathbb{R}^n)$.) Using Lemma 2.1 and Lemma 2.2 it follows that $\hat{\psi} * f \equiv 0$. This gives us the desired contradiction because $\hat{\psi} * f$ is non-trivial!

We are now in a position to state and prove our main result.

THEOREM 3.2: *Let E be a bounded Borel set in $\mathbb{R}^n$, with positive Lebesgue measure. Then P_E is injective on $L^p(\mathbb{R}^n)$ if $1 \leq p \leq 2n/(n-1)$.*

Proof: Let $1 \leq p \leq 2n/(n-1)$ and let $X = \{f \in L^p(\mathbb{R}^n) : P_E f = 0\}$. Then it is easy to show that $f \in X$ if and only if $f * \check{1}_{TE} = 0$ for all $T \in \mathrm{SO}(n)$, where $\check{1}_A(x) = 1_A(-x) = 1_{-A}(x)$. From this it follows easily that X is a closed subspace which is moreover closed under translations and rotations. Suppose $X \neq (0)$. Using the above observations it is easy to show that there exists a non-trivial $f \in X$, f continuous. Thus $f * \check{1}_{TE} = 0$ for all $T \in \mathrm{SO}(n)$ and it will follow that $f * \check{1}_E^{\#} = 0$. But $\check{1}_E^{\#}$ is a non-trivial, compactly supported, radial distribution and hence by the previous proposition $f \equiv 0$, which gives us a contradiction. Thus $X = (0)$ and the proof of the theorem is complete.

4. Some Wiener–Tauberian type results

It is easy to see that the condition P_E is injective on $L^p(\mathbb{R}^n)$, $1 < p \leq 2n/(n-1)$ is equivalent (by duality) to the condition that span $\{{}^g 1_E : g \in M(n)\}$ is dense in

$L^q(\mathbb{R}^n)$ for $2n/(n+1) \leq q < \infty$. For any function f and any $g \in M(n)$, by ${}^g f$ we mean the function ${}^g f(x) = f(g.x), x \in \mathbb{R}^n$. (Note that we are excluding the case $p = 1$ in the above statement because $(L^\infty)^* \neq L^1$.) This denseness condition is a 'Wiener–Tauberian type' statement and therefore a natural question to ask is: If $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, what conditions on the Fourier-transform of f will be equivalent to span $\{{}^g f : g \in M(n)\}$ being dense in $L^q(\mathbb{R}^n)$? The following theorem answers this question for a range of values of q, $1 \leq q < \infty$, and gives sufficient conditions for the remaining values of q. We mainly use ideas from the proof of Proposition 3.1 We also use the classical Wiener–Tauberian Theorem in certain cases and the fact that if f is a *radial* function in $L^p(\mathbb{R}^n)$, $1 \leq p < 2n/(n+1)$, then $\hat{f}$ is actually given by a continuous function on $\mathbb{R}^n \backslash \{0\}$. The proof of the latter fact is briefly sketched in the proof of Theorem 4.1.

(Analogues of Wiener's theorem, even in the one-dimensional case, when $p \neq 1$ or 2 are quite hard. Therefore it is quite surprising that for $L^p(\mathbb{R}^n), n \geq 2$, and with the rigid motion group instead of the translation group, we are able to get reasonably complete results!)

Before stating the theorem we set up some notation.

For $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n), 1 \leq q \leq \infty$, let $S = \{r > 0 : \hat{f} \equiv 0 \text{ on } M_r\}$ and $X = \text{span}\{{}^g f : g \in M(n)\}$. Clearly S is a closed subset of $\mathbb{R}^+$.

THEOREM 4.1:

(1) *Let $f \in L^1(\mathbb{R}^n)$. Then X is dense in $L^1(\mathbb{R}^n)$ if and only if $\hat{f}(0) \neq 0$ and S is empty.*

(2) *Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n), 1 < q < 2n/(n+1)$. Then X is dense in $L^q(\mathbb{R}^n)$ if and only if S is empty.*

(3) *Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n), 2n/(n+1) \leq q < 2$. If every point of S is an isolated point, then X is dense in $L^q(\mathbb{R}^n)$.*

(4) *Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n), 2 \leq q \leq 2n/(n-1)$. If S is of zero measure in $\mathbb{R}^+$, then X is dense in $L^q(\mathbb{R}^n)$.*

(5) *Let $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n), 2n/(n-1) < q < \infty$. Then X is dense in $L^q(\mathbb{R}^n)$ if and only if S is nowhere dense.*

Proof: (1) The proof follows easily from the classical Wiener–Tauberian theorem. (See Proposition 9.4 in [7].)

(2) Suppose X is dense in $L^q(\mathbb{R}^n), 1 < q < 2n/(n+1)$. If there exists an r in S, then the radial function $\phi_r \in L^p(\mathbb{R}^n)$ where $1/p + 1/q = 1$, since $p > 2n/(n-1)$.

Using the fact that $\hat{f}$ vanishes on M_r and that ϕ_r is radial, it can be proved that $\int {}^g f \phi_r = 0$ for all $g \in M(n)$ which contradicts the fact that X is dense in L^q. Hence S is empty.

Conversely, assume S is empty. Suppose X is *not* dense in L^q. Then there exists a non-trivial $h \in L^p(\mathbb{R}^n), 1/p + 1/q = 1, 2n/(n-1) < p < \infty$ such that $\int {}^g f h = 0, \forall g \in M(n)$. Then, as argued earlier in section 3, we can assume h to be smooth and radial. It will follow that $h * f \equiv 0$. Convolving h with a smooth compactly supported approximate identity we can even assume that h is *bounded.* Since f is no longer assumed to be compactly supported, $\hat{f}$ may not be smooth and hence $\hat{f}\hat{h}$ may not make sense! However by Theorem 9.3 of [7] which is essentially the Wiener–Tauberian Theorem in disguise, we can still conclude that $\operatorname{Supp} \hat{h} \subseteq Z_{f^\wedge}$. (See also Proposition 6.1 in [6].) Since h is radial and S is empty, 0 can be the only possible point in $\operatorname{Supp} \hat{h}$. If $\operatorname{Supp} \hat{h} = \{0\}$, then h is a non-trivial polynomial and this is impossible, since h is also in L^p with $p < \infty$. Therefore, $\operatorname{Supp} \hat{h}$ is empty. Hence $h \equiv 0$, a contradiction.

(3) Suppose X is not dense in $L^q(\mathbb{R}^n), 2n/(n+1) \leq q < 2$. Then as argued in (2) above there exists a non-trivial smooth radial *bounded* $h \in L^p(\mathbb{R}^n), 1/p + 1/q = 1$ with $2 < p \leq 2n/(n-1)$ such that $h * f = 0$. As before this implies $\operatorname{Supp} \hat{h} \subseteq Z_{f^\wedge}$. Also, as in the Proposition 3.1, we have $\alpha > 0$ such that $M_\alpha \subseteq \operatorname{Supp} \hat{h} \subseteq Z_{f^\wedge}$. But then $\alpha \in S$ and each point of S is an isolated point. Therefore there exists an $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha + \epsilon) \cap S = \{\alpha\}$. Consider the annulus

$$A_\epsilon = \{x \in \mathbb{R}^n \colon \alpha - \epsilon < \|x\| < \alpha + \epsilon\}.$$

Choose $\psi \in \mathcal{D}(\mathbb{R}^n)$ as in the proof of Proposition 3.1. Then $\operatorname{Supp} \psi \hat{h} = M_\alpha$. Equivalently $\operatorname{Supp}(\hat{\psi} * h)^\wedge = M_\alpha$. Using Lemma 2.1 and Lemma 2.2 and the fact that $\hat{\psi} * h \in L^p(\mathbb{R}^n), 2 < p \leq 2n/(n-1)$, we conclude that $\hat{\psi} * h \equiv 0$ exactly as in the proof of Proposition 3.1. But then this contradicts that $\operatorname{Supp}(\hat{\psi} * h)^\wedge = M_\alpha$.

(4) Suppose X is not dense in $L^q(\mathbb{R}^n), 2 \leq q \leq 2n/(n-1)$. Then, as before, there exists $h \in L^p(\mathbb{R}^n), 1/p + 1/q = 1$, a non-trivial radial function such that $h * f = 0$. Since $2n/(n+1) \leq p \leq 2, \hat{h}$ is defined as a function. Therefore $\hat{f}\hat{h} = 0$ and this together with the fact that S is of zero measure in $\mathbb{R}^+$ implies that $\hat{h}$ is zero a.e. in $\mathbb{R}^n$. But then $h = 0$ a.e., a contradiction.

(Note that for $q = 2$, it is enough to take $f \in L^2(\mathbb{R}^n)$ together with the condition that S is of zero measure in $\mathbb{R}^+$. In fact, this condition is also necessary in this case. This follows easily from the Plancherel Theorem.)

(5) Assume X is dense in $L^q(\mathbb{R}^n)$, $2n/(n-1) < q < \infty$. Since S is closed in $\mathbb{R}^+$, the fact that S is nowhere dense is equivalent to saying that S does not contain any non-empty open interval . So if S is not nowhere dense, then $\hat{f}$ vanishes on some annulus $A_{r_1,r_2} = \{x \in \mathbb{R}^n : r_1 < \|x\| < r_2\}, r_2 > r_1 > 0$. Choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ non-trivial and radial such that $\operatorname{Supp} \hat{\psi} \subseteq A_{r_1,r_2}$. Hence $\hat{f}\hat{\psi} = 0$ i.e., $f * \psi = 0$. Also $\psi \in L^p(\mathbb{R}^n), 1/p + 1/q = 1$. Since ψ is radial, this implies that $\int {}^g f \psi = 0$ for all $g \in M(n)$. This contradicts the fact that X is dense in L^q.

To prove the 'if' part we need the following observation : For a *radial* $h \in L^p(\mathbb{R}^n)$, $1 \leq p < 2n/(n+1), \hat{h}$ is given pointwise on $\mathbb{R}^n \setminus \{0\}$ by the following expression:

$$\hat{h}(y) = \int_{\mathbb{R}^n} h(x) \phi_{\|y\|}(x) dx, \quad y \in \mathbb{R}^n \setminus \{0\}.$$

As $\phi_\lambda \in L^q(\mathbb{R}^n), \forall q > 2n/(n-1)$, $\lambda > 0$, the integral on the right hand side makes sense. Using the asymptotic behaviour of $\phi_\lambda, \lambda \in \mathbb{R}^+$, and the dependence of the behaviour with respect to the parameter λ, we can show that $\lambda \to \int_{\mathbb{R}^n} h(x)\phi_\lambda(x)dx$ is a continuous function on $\mathbb{R}^+$. Hence $\hat{h}$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.

Coming back to the proof of 'if' part in (5), suppose X is not dense in L^q. Then as before there exists a non-trivial radial $h \in L^p(\mathbb{R}^n), 1/p + 1/q = 1$, $1 < p < 2n/(n+1)$, with $h * f = 0$. Then $\hat{f}\hat{h} \equiv 0$ on $\mathbb{R}^n \setminus \{0\}$. Since by assumption, S is nowhere dense, for any $M_a, a > 0$, there are points as close to M_a as we like where $\hat{f}$ does not vanish. Therefore $\hat{h}$ vanishes on these points. Since $\hat{h}$ is radial this implies that it vanishes on spheres arbitrarily close to M_a. By continuity of $\hat{h}$ in $\mathbb{R}^n \setminus \{0\}$ we conclude that $\hat{h} = 0$ on $M_a, a > 0$ i.e., $\hat{h} \equiv 0$ on $\mathbb{R}^n \setminus \{0\}$. But then $h = 0$ a.e., a contradiction.

Since we are considering L^p-functions we need not have confined ourselves to bounded Borel sets in Section 3. We could as well have considered Borel sets of finite positive measure. In that case what is the analogue of Theorem 3.2 ?

The following corollary to Theorem 4.1 answers this question:

COROLLARY 4.2: *Let E be a Borel subset of $\mathbb{R}^n$ of finite positive measure. Let $S = \{r > 0 : \hat{1}_E \equiv 0$ on $M_r\}$.*

(1) *P_E is injective on $L^p(\mathbb{R}^n)$, $1 \leq p < 2n/(n+1)$, if and only if S is nowhere dense.*

(2) *If S is of measure zero, then P_E is injective on $L^p(\mathbb{R}^n), 2n/(n+1) \leq p \leq 2$.*

(3) *If every point of S is an isolated point, then P_E is injective on $L^p(\mathbb{R}^n)$, $2 < p \leq 2n/(n-1)$.*

(4) *P_E is injective on $L^p(\mathbb{R}^n), 2n/(n-1) < p \leq \infty$, if and only if S is empty.*

Proof: All the statements above, except for $p = 1$, follow by duality. The case $p = 1$ can be proved quite easily (and is well-known). The case $p = \infty$ has been already proved in [6], Corollary 6.3.

(Note that a bounded Borel set of positive measure satisfies the conditions (1), (2) and (3) of Corollary 4.2.)

In the case $p = 2$, the condition 'S is of zero measure' is both a necessary and sufficient condition for P_E to be injective.

5. Concluding remarks

Let Y be a symmetric space of non-compact type. Then one can modify the main result in [10] to show the following :

PROPOSITION 5.1: *If E is a Borel subset of Y of finite positive measure (with respect to the canonical measure on Y), then P_E is injective on $L^p(Y), 1 \leq p \leq 2$.*

In view of the proposition in [8], this is the best possible result in general (i.e. without assuming anything about the 'shape' of E). Thus the behaviour for Euclidean spaces is slightly different from that for symmetric spaces — the main reason for this being the difference in the asymptotic behaviour of the corresponding 'elementary' spherical functions. In fact, one has the following interesting result from which Proposition 5.1 also follows :

PROPOSITION 5.2: *Let $0 \neq f \in L^1(Y) \cap L^q(Y), 2 \leq q < \infty$. Then the span of $\{{}^g f : g \in G\}$ is dense in $L^q(Y)$. (Here G is the connected component of the group of isometries of Y containing the identity.)*

We omit the proof of this. For the case $p = 2$ this observation has been made in [11]. The case $f \in L^p \cap L^1, p > 2$ can be proved in a similar way.

Just to complete the story, we finally look at the case of symmetric spaces of compact type. The following result, though not stated in this fashion, is implicit in [4] (for any unexplained terminology in the proposition see [4]):

PROPOSITION 5.3: *Let $Y = G/K$ be a symmetric space of compact type (where G is a compact, connected, semi-simple Lie group and K a suitable closed subgroup of G). Let $f \in L^p(Y)$, $1 \leq p < \infty$. Then* $\mathrm{Span}\{{}^g f : g \in G\}$ *is dense in*

$L^p(Y)$ if and only if $\pi(f) \neq 0$ for each class -1, irreducible, unitary representation π of G. (Here we view a function on Y as a function on G invariant under the right action of K.)

Note added in proof: For an analogue of Wiener's theorem for $L^p(\mathbb{R})$, $p \neq 1$ or 2, we refer the reader to the work of A. Beurling. (See, for example, the concluding portion of Section 58 in: W. F. Donoghue, *Distributions and Fourier Transforms*, Academic Press, New York, 1969.) We thank G. B. Folland for drawing our attention to this.

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